The Power of Asymmetry in Binary Hashing

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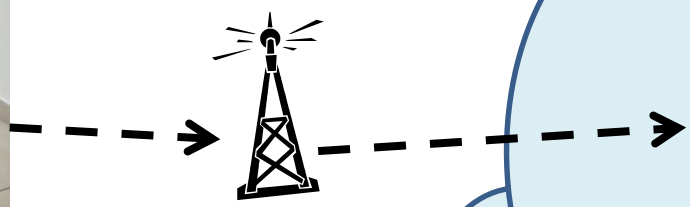

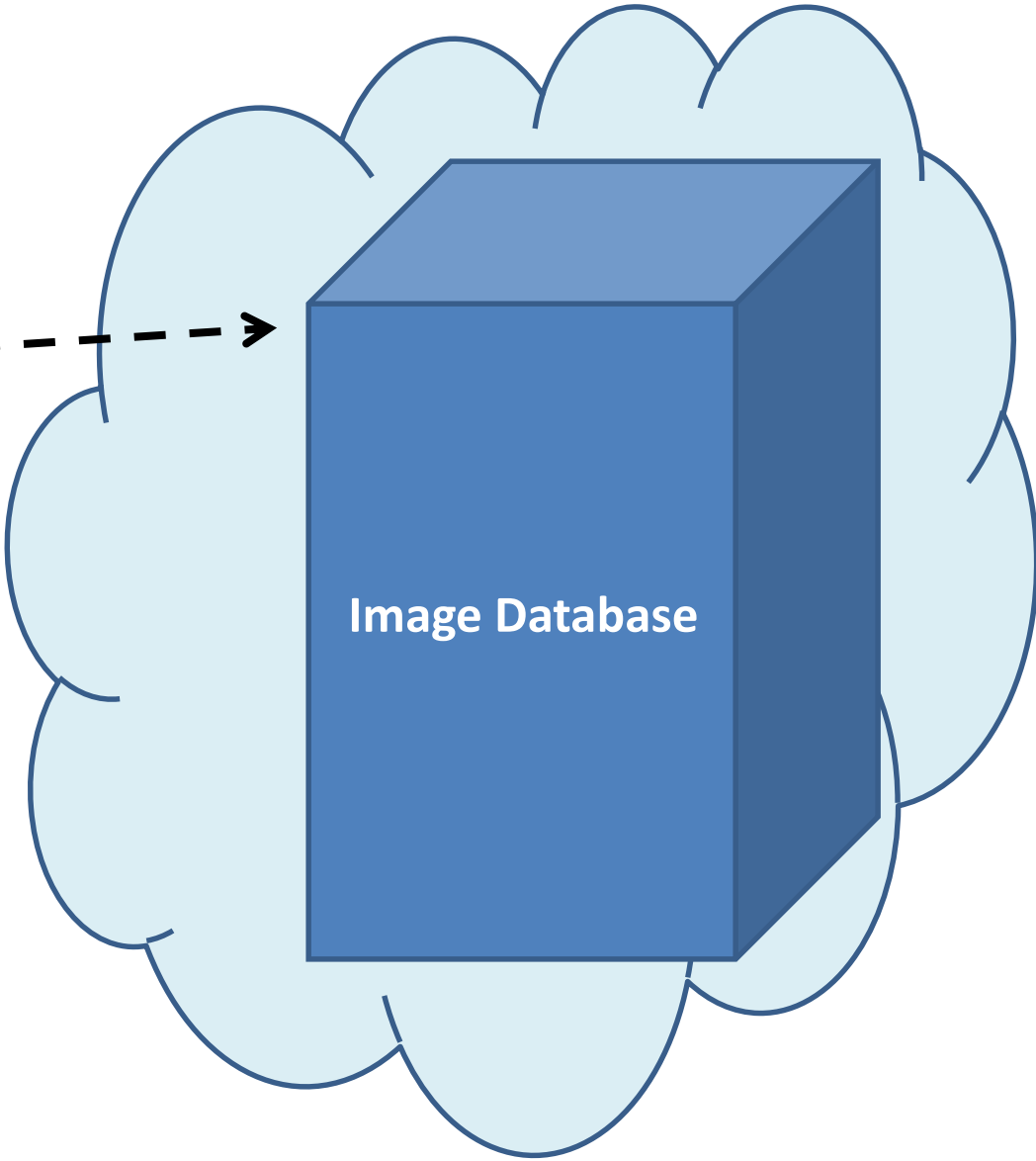

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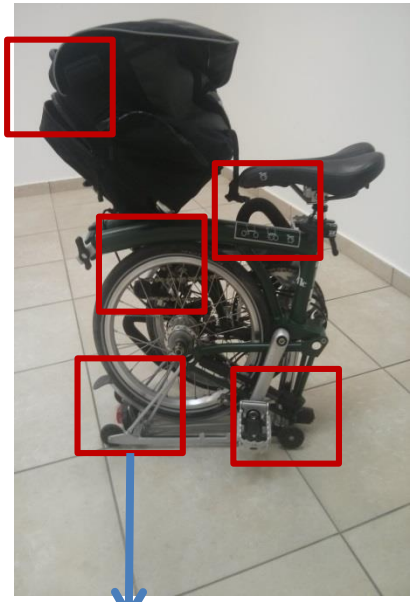

f( ) = 0110100011110

1010100111100

1011101001101

Database of hashed image patches

0001111001001

f( ) = 011011001**0**110

Binary Hashing

$$[\, d_{hamming}(\, \underset{\cup}{f}(x), \underset{\cup}{f}(x')\,) < \theta\,] \approx \underset{\cup}{Sim}(x,x')$$

$\underset{\cup}{f}(x) \in \{\pm 1\}^k$ $\quad \underset{\cup}{f}(x') \in \{\pm 1\}^k$ $\quad \underset{\cup}{Sim}(x,x') \in 0/1$

Binary Hashing

$$[\, d_{hamming}(\, \underset{\cup}{\{\pm 1\}^k}\ f(x), \underset{\cup}{\{\pm 1\}^k}\ g(x')\,) < \theta\,] \approx \underset{\cup}{0/1}\ Sim(x,x')$$

Even if Sim(x,x') is symmetric and "well behaved":

- Use **f**(x) to hash objects in the database
- Use **g**(x) to hash queries

- No additional memory, communication or computation to perform query
- No increase in hash table size or lookup complexity
- We will show: **shorter bit length; higher accuracy**

Outline

- **Theoretical Observation:**

Capturing similarity with arbitrary binary hashes

- **Empirical Investigation:**

Database lookup with linear threshold hashes

Capturing Similarity with an Arbitrary Binary Code

- Given similarity S(x,x') over objects $\mathcal{X}=\{x_1,x_2,\ldots,x_n\}$, want mapping $f:\mathcal{X}\rightarrow\pm 1^{\boldsymbol{k}}$, s.t.

$$S_{ij} = S(x_i,x_j) = [d(f(x_i),f(x_j))<\theta]$$

- $f(\cdot)$ arbitrary, specified by $u_1,\ldots,u_n$, $u_i=f(x_i)$
- What is shortest ***k*** possible?

$$\exists\, u_1,\ldots,u_n\in\pm 1^{\boldsymbol{k}},\ \theta\in\mathbb{R} \quad \forall_{ij}\ S_{ij}=[d(u_i,u_j)<\theta]$$

- What is shortest ***k*** if we allow asymmetry?

$$\exists\, u_1,\ldots,u_n,v_1,\ldots,v_n\in\pm 1^{\boldsymbol{k}},\ \theta\in\mathbb{R} \quad \forall_{ij}\ S_{ij}=[d(u_i,v_j)<\theta]$$

$v_j=g(x_j)$

The Power of Asymmetry for Arbitrary Binary Hashes

Theorem: For any r, there exists a set of points in Euclidean space, s.t. to capture

$$S(x_i,x_j) = [\ |x_i-x_j| < 1\ ]$$

using a symmetric binary hash we need $k_{sym} \geq 2^{r-1}$ bits, but using an asymmetric hash, we need only $k_{asym} \leq 2r$ bits.

$$X^T X = \frac{1}{n}\begin{bmatrix} n & -1 & \cdots & -1 & 1 & 1 & \cdots & 1 \\ -1 & n & \cdots & -1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ -1 & -1 & \cdots & n & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & n & -1 & \cdots & -1 \\ 1 & 1 & \cdots & 1 & -1 & n & \cdots & -1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & -1 & -1 & \cdots & n \end{bmatrix} \qquad n=2^r$$

Binary Hashing as Matrix Factorization

$$[\, d_{hamming}(\, u_i,\, u_j) < \theta \,] \approx \text{Sim}(x,x')$$

Binary Hashing as Matrix Factorization

$$[\ \langle u_i, u_j \rangle < \theta\ ] \approx \text{Sim}(x,x')$$

Given similarity matrix $\mathbf{S} \in \{\pm 1\}^{n \times n}$:

min k s.t. $U \in \{\pm 1\}^{k \times n}$ symmetric

$\text{sign}(U^T U - \theta) = \mathbf{S}$

min k s.t. $U, V \in \{\pm 1\}^{k \times n}$ asymmetric

$\text{sign}(U^T V - \theta) = \mathbf{S}$

Bonus: Approximation Algorithm via SDP Relaxation

Given similarity matrix $\mathbf{S} \in \{\pm 1\}^{n \times n}$:

min k s.t. $U \in \{\pm 1\}^{k \times n}$ symmetric

$\text{sign}(Y - \theta) = \mathbf{S}$

$Y = U^T U$

min k s.t. $U, V \in \{\pm 1\}^{k \times n}$ asymmetric

$\text{sign}(Y - \theta) = \mathbf{S}$

$Y = U^T V$

Bonus: Approximation Algorithm via SDP Relaxation

Given similarity matrix $\mathbf{S} \in \{\pm 1\}^{n \times n}$:

min k s.t. $\|U_i\| \leq \sqrt{k}$ symmetric

$\boldsymbol{S}_{ij}(Y_{ij} - \theta) \geq 1$

$Y = U^T U$

min k s.t. $\|U_i\|, \|V_j\| \leq \sqrt{k}$ asymmetric

$\boldsymbol{S}_{ij}(Y_{ij} - \theta) \geq 1$

$Y = U^T V$

Bonus: Approximation Algorithm via SDP Relaxation

Given similarity matrix $\mathbf{S} \in \{\pm 1\}^{n \times n}$:

$\min \| Y \|_{\max}$ s.t. symmetric

$$\boldsymbol{S}_{ij}\left(Y_{ij} - \theta\right) \geq 1$$

$$Y \succ 0$$

$\min \| Y \|_{\max}$ s.t. asymmetric

$$\boldsymbol{S}_{ij}\left(Y_{ij} - \theta\right) \geq 1$$

$$\|Y\|_{\max}^2 = \min_{Y=LR} \|L\|_{\infty,2} \|R\|_{\infty,2}$$

SDP Relaxation—Rounding

$$\|Y\|_{\max}^2 = \min_{Y=LR} \|L\|_{\infty,2} \|R\|_{\infty,2}$$

Using random vectors $z_1,\ldots,z_k$:

$u_i = \text{sign}(L_i^T z_i)$

$v_i = \text{sign}(R_i^T z_i)$

$k = O((k_{opt})^2 \log n)$

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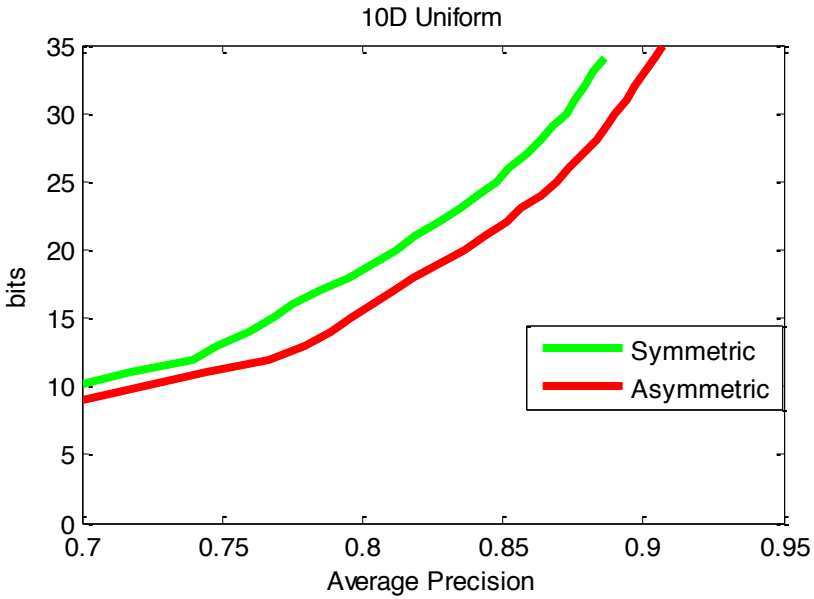
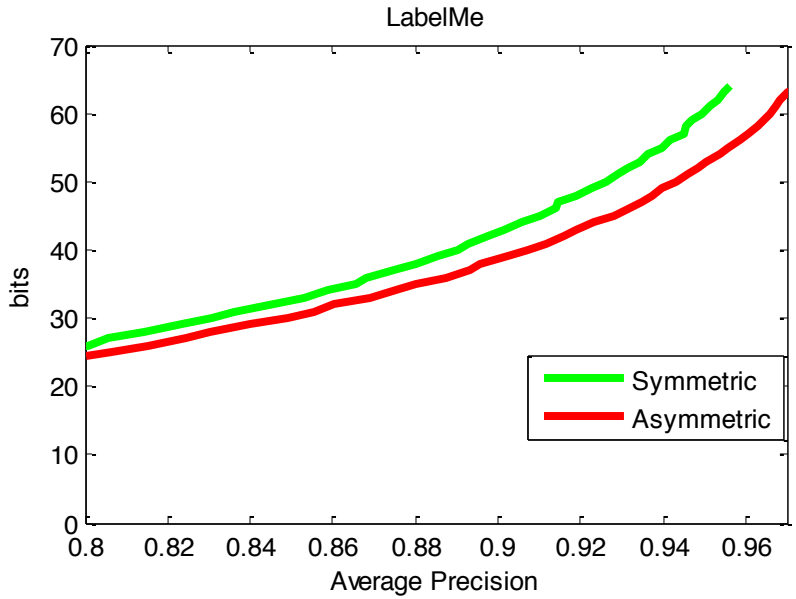
Arbitrary Binary Hashes: Empirical Evaluation





Parametric Mappings

- **f**(x) = $\phi_{\boldsymbol{F}}$(x) has some parametric form
  - E.g. $\phi_{\boldsymbol{F}}$(x) = sign(***F***x), ***F***$\in R^{k\times d}$
  - Could be more complex, e.g. multi-layer network, kernel based, etc.
- Why restrict **f**(·)?
  - Generalization: learn ***F*** using objects $x_1,\ldots,x_n$, then receive new objects x as queries
  - Compactness of representation

- Asymmetric extension:

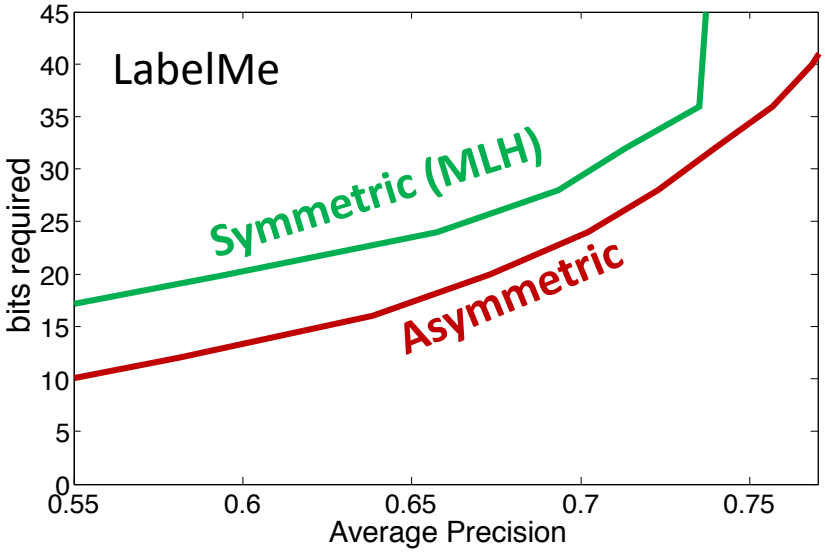
$$S(x,x') \approx [\, d(\phi_{\boldsymbol{F}}(x)\,,\, \phi_{\boldsymbol{G}}(x')\,) < \theta\,]$$

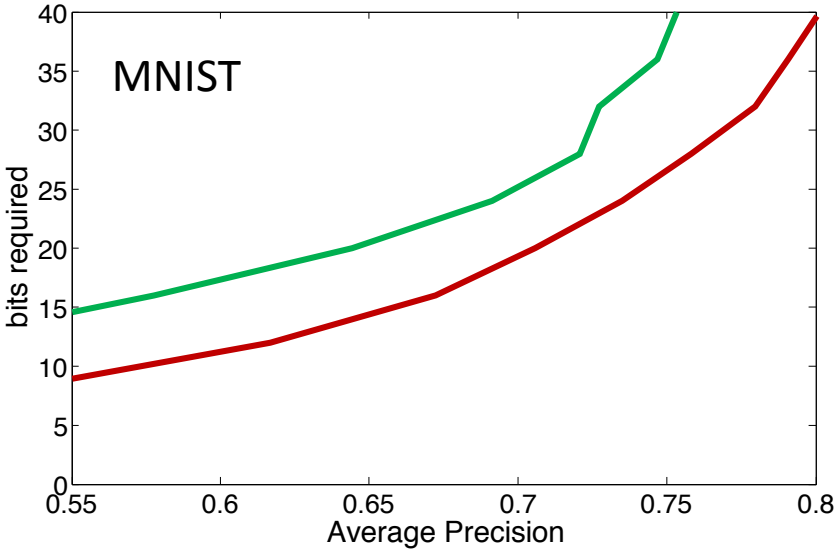
  - Learn params ***F***, ***G*** from training data (= pairs of database objects)
  - Hash database objects using $\phi_{\boldsymbol{G}}$
  - Hash queries using $\phi_{\boldsymbol{F}}$

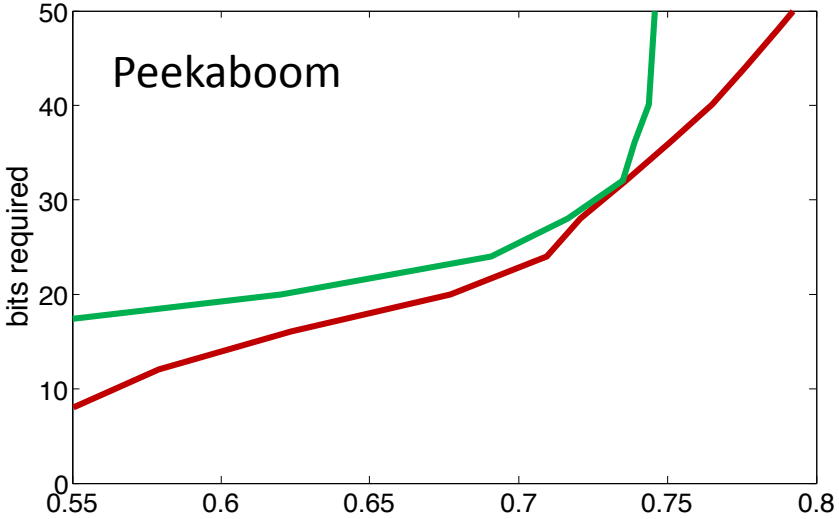
A bit on optimization

- Loss function : L(ϕ_F(x), ϕ_G(x'), S(x,x'))
- Parameters F, G
- For ϕ_F(x) = sign(Fx):
  - Updating single row of F (responsible for single bit in the hashing) entails solving a single weighted binary classification problem

Empirical Results using Asymmetric Linear Threshold Hashes





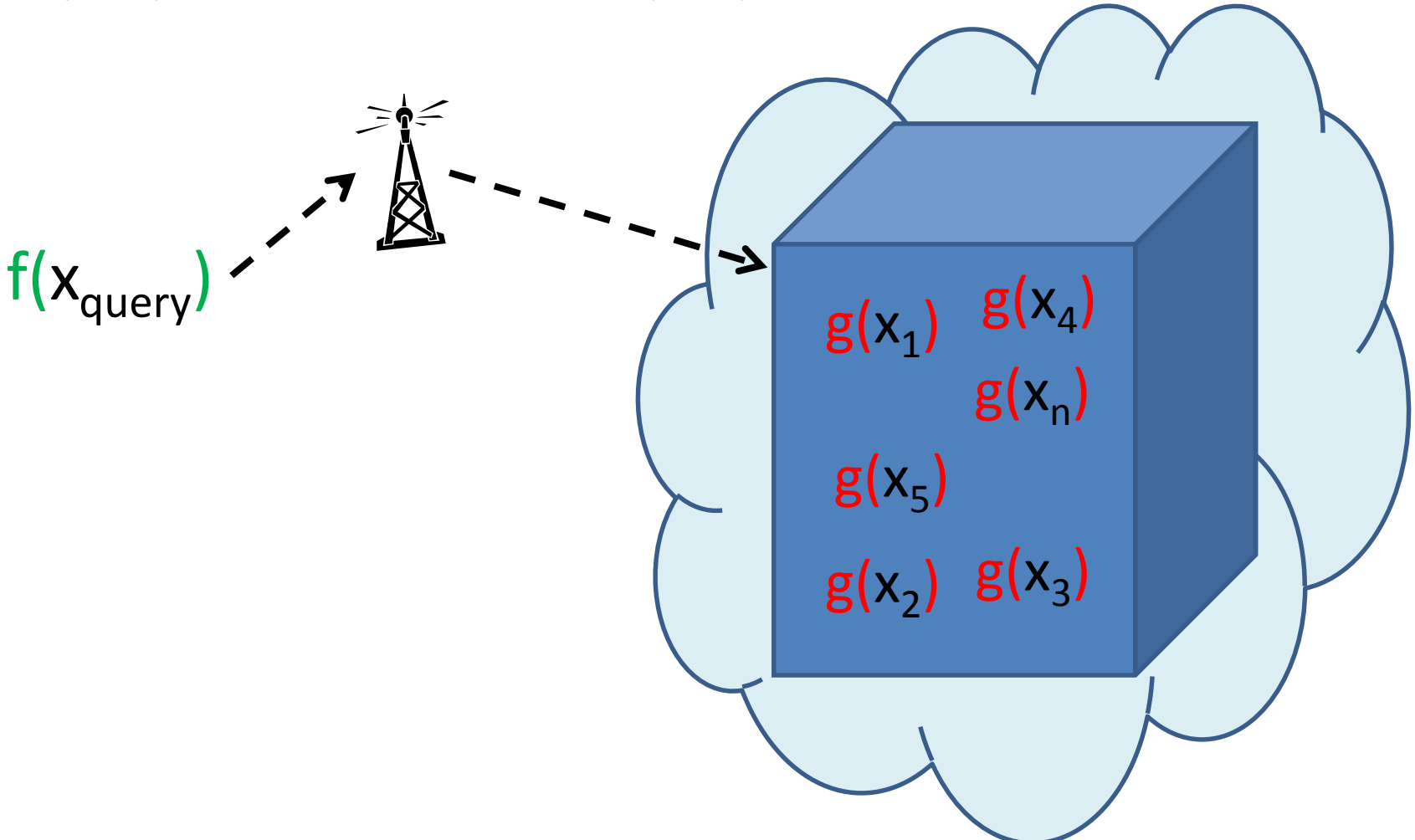


Comparison with learning symmetric linear threshold hashes using Minimum Loss Hashing [Norouzi Fleet ICML 2011]

Training on all pairs of database objects, average precision measured on held out query objects.

Even More Power: Searching a Static Database

$$S(x_{query}, x_i) \approx [\, d(\, f\,(x_{query})\,,\; g\,(x_i)\,) < \theta\,]$$



Even More Power: Searching a Static Database

$$S(x_{query}, x_i) \approx [\, d(\phi_F(x_{query}), \quad v_i \quad) < \theta \,]$$

- No need to generalize: only used on database objects
- Stored in database hash, no need for compactness
- **Can use arbitrary mapping $g(x_i)=v_i$**

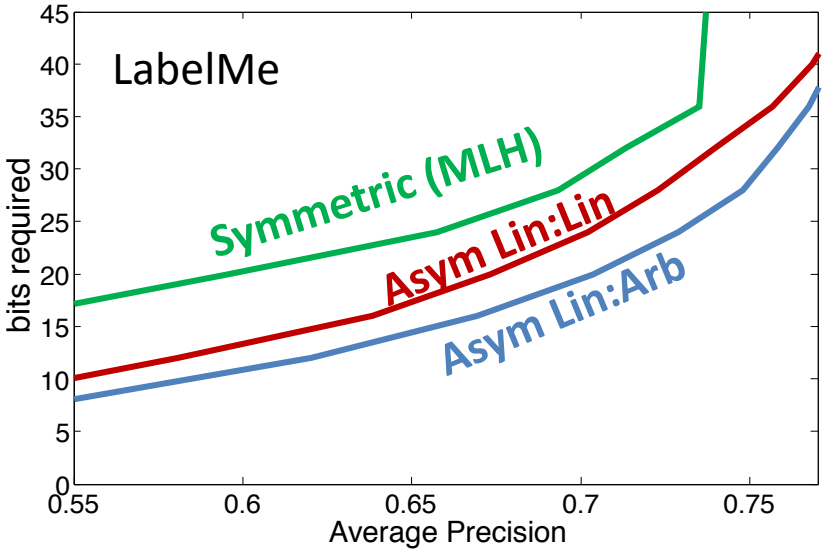
- Learn parametric $\phi_F(\cdot)$ and arbitrary vectors $v_1,\ldots,v_n$ such that on $x_1,\ldots,x_n$:

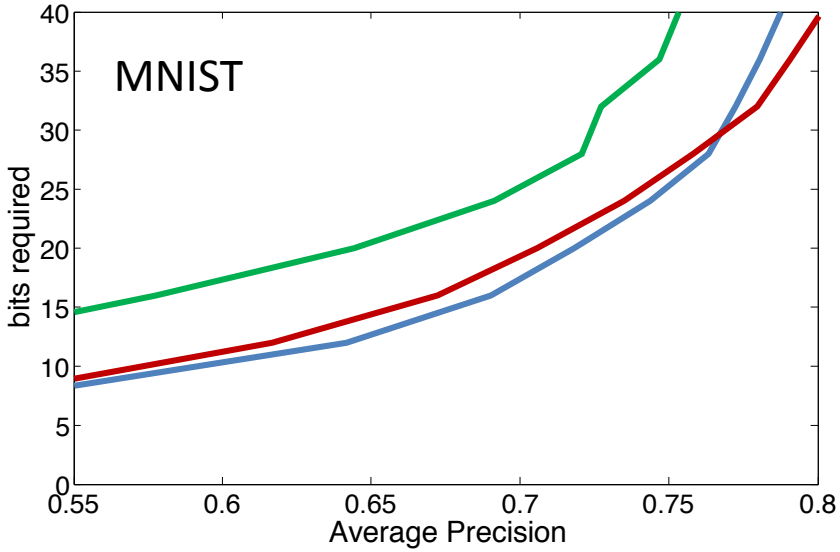
$$S(x_i,x_j) \approx [\, d(\phi_F(x_i), v_j) < \theta \,]$$

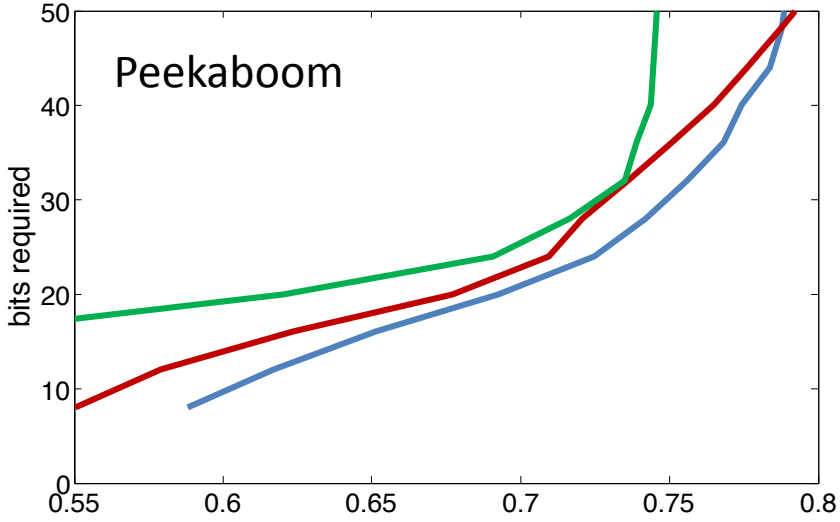
- For query x, use $d(\phi_F(x),v_j)$ to aprox $S(x,x_j)$

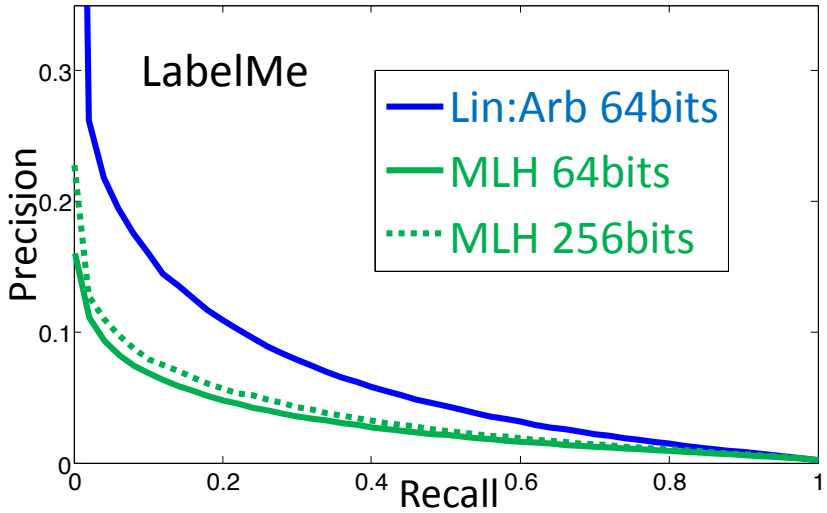
  e.g. find v_j in DB with small $d(\phi_F(x),v_j)$

Empirical Results using Linear-Threshold-to-Arbitrary Hashes









Summary

Neyshabur et al, "The Power of Asymmetry in Binary Hashing", NIPS 2013

- Binary Hashing
  - Fast similarity approximation
  - Approximate retrieval
  - Nearest Neighbor search
  - Locality Sensitive Hashing (LSH)
- In many applications: want short bit-length
  - Smaller hash tables
  - Super-fast hamming distance eval on short words
  - Fewer bits to transmit

- **Asymmetric hashes can enable better approximation with shorter bit-length!**
  - Even if similarity function symmetric and well-behaves
  - In most applications: no additional computational or memory costs